AN INTRODUCTION TO BRST AND BV FORMALISM

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1. Very brief introduction

These are some brief lecture notes corresponding to a small minicourse delivered at the Student Colloquium and School on Mathematical Physics in Stará Lesná in 2024. They cover the basics of the BRST and BV formalism, which corresponds to an efficient and widely applicable framework for quantisation of classical field theories. To the largest part the exposition is based on the treatment in Weinberg's Quantum theory of fields (vol 2), so we refer the interested reader to this particular source. I am also very grateful to Pavol Ševera for discussions and clarifications.

We will work in finite dimensional setups — this is solely in order to save time (by avoiding writing integrals) and declutter the notation. Alternatively, one can say that we use the De Witt notation.

2. BRST formalism

2.1. **The task.** Stories often start with a hero setting out for a difficult task. In this story we are the heroes and our task is the following.

Suppose we have a compact and connected finite-dimensional Lie group G (think of it as the infinite-dimensional group of gauge symmetries) acting on the manifold $\mathbb{R}^n$ (a.k.a. the space of fields) and preserving the measure $\mathcal{D}x = dx^1 \dots dx^n$ (the path integral measure). We will denote the fundamental vector fields by e_a. We were entrusted with the mission of calculating the integral (path integral of a gauge invariant theory with insertions of gauge-invariant operators)

$$\int_{\mathbb{R}^n} \psi(x) e^{iS(x)} \mathcal{D}x,$$

where both ψ and S are G-invariant functions on $\mathbb{R}^n$.

Since we are (pretending to be) physicists, we want to calculate this quantity using perturbation techniques. In order to do this, we need a non-degenerate critical point of S (so that we can get the propagator by inverting the quadratic term in S). However, this is not possible — the G-invariance of S implies that any critical point is degenerate. What shall we do?

2.2. **Faddeev–Popov trick.** We note that this problem is related to the fact that we are unnecessarily performing integrations along the orbits of the group action. Since we want to regard the G-related points as gauge equivalent, it would suffice to count each orbit only once, i.e. make the integration run over some "submanifold" $\mathcal{S} \subset \mathbb{R}^n$ which intersects each orbit once.[1] Suppose we have such an $\mathcal{S}$, given by the vanishing of a set of functions $f_a(x)$ (gauge fixing conditions). Then we have[2]

$$\int_{\mathbb{R}^n} \psi e^{iS} \mathcal{D}x \sim \int_{\mathbb{R}^n} \mathcal{D}x \, \psi e^{iS} \prod_a \delta(f_a(x)) \det A, \qquad A_{ab}(x) := e_a(f_b).$$

[1] All notes related to BRST, Faddeev–Popov method, or gauge fixing procedures should mention *Gribov ambiguity* — so we are mentioning it here.

[2] We will use $\sim$ to denote equality up to a nonzero constant multiple.

To see the need for the determinant correction, note that for instance both $\delta(2f)$ and $\delta(f)$ have the effect of restricting the integration to the vanishing locus of f but they are not equal (in fact $\delta(2f) = \frac{1}{2}\delta(f)$).

This looks bad! We have just obtained two relatively nasty extra factors in our path integral! Why is this better? The trick is that we can actually represent both of these ingredients as further integrals of convenient exponentials, thus effectively modifying S. Explicitly, we can introduce new fermionic variables c^a, b^a (ghosts) and new bosonic variables h^a (Nakanishi–Lautrup fields) and write

$$\int_{\mathbb{R}^n} \psi(x) e^{iS(x)} \mathcal{D}x \sim \int \mathcal{D}x\, \mathcal{D}b\, \mathcal{D}c\, \mathcal{D}h\, \psi(x) e^{i(S(x)+b^a A_{ab}(x) c^b + h^a f_a(x))}. \tag{1}$$

2.3. **A variant.** One can modify the argument slightly and instead of inserting "$\prod_a \delta(f_a(x)) \det A$" into the integral we can insert "$\prod_a \delta(f_a(x) - \nu_a) \det A$", for some numbers ν_a (after all, the argument should be independent of the precise form of the gauge-fixing function). Or we can insert a linear combination

$$\sum_\beta w_\beta \prod_a \delta(f_a(x) - \nu_a^\beta) \det A,$$

as long as $\sum_\beta w_\beta = 1$. Or we can take a particular continuous linear combination, a.k.a. integral, i.e. insert

$$\int \mathcal{D}\nu\, w(\nu) \prod_a \delta(f_a(x) - \nu_a) \det A = w(f(x)) \det A,$$

with $\int \mathcal{D}\nu\, w(\nu) = 1$. Since we like (or at least know what to do with) exponentials, we will pick the weight function w to be Gaussian. We thus obtain

$$\int_{\mathbb{R}^n} \psi e^{iS} \mathcal{D}x \sim \int \mathcal{D}x\, \mathcal{D}b\, \mathcal{D}c\, \psi e^{i(S(x)+b^a A_{ab}(x) c^b + f^a(x) f_a(x))}.$$

The moral of either method is that we now resolved our original problem by replacing the theory by a larger one (with more fields) which however has a nondegenerate critical point, and so we can now happily proceed with quantisation.

2.4. **BRST symmetry.** Returning back to (1) we see that our new action

$$S'(x,b,c,h) = S(x) + bA(x)c + hf(x) \tag{2}$$

is no longer gauge invariant. This is of course good, since we wanted to get rid of the troublesome gauge invariance. However, as we will see in a moment there is a new, so called BRST, symmetry emerging. It is given by the *BRST* (or sometimes called *Slavnov*) *operator* (i.e. a vector field on our newly extended space of fields)

$$Q = c^a e_a^i(x) \partial_{x^i} - \tfrac{1}{2} f^a{}_{bc} c^b c^c \partial_{c^a} - h^a \partial_{b^a},$$

where $f^a{}_{bc}$ are the structure constants of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Note that this is an odd operator (when acting on a bosonic function it gives a fermionic answer, and vice versa). It also has the amazing property

$$Q^2 = 0.$$

Finally, if we assign a degree (ghost number) to our variables according to

$$\deg x^i = 0, \qquad \deg c^a = 1, \qquad \deg b^a = -1, \qquad \deg h^a = 0,$$

then we get that $\deg S' = \deg S = 0$ and $\deg Q = 1$ (i.e. Q raises the degree by 1).

Let us now see why $QS' = 0$. First, $QS = c^a(e_a S) = 0$ due to our original G-invariance of S. The invariance of the remainder follows from the fact that

$$b^a A_{ab}(x) c^c + h^a f_a(x) = -Q(b^a f_a(x)) \tag{3}$$

and the nilpotency of Q.

Let us list some further amazing facts/observations:

Observation 1. As noted above, G-invariant functions $\psi(x)$ automatically satisfy $Q\psi = 0$. In fact, more is true: *The cohomology of Q in degree zero is isomorphic to the space of G-invariant functions $\psi(x)$.* In other words, for any degree 0 function $\Psi(x, c, b, h)$ satisfying $Q\Psi = 0$ there exists a unique G-invariant $\psi(x)$ such that

$$\Psi(x, c, b, h) = \psi(x) + Q\,\Xi(x, c, b, h)$$

for some function Ξ of degree -1.

Observation 2. The measure $\mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h$ is also Q-invariant. In particular, for any function $\Xi(x, b, c, h)$ we have

$$\int \mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h\,(Q\Xi)e^{iS'} = 0.$$

Similarly, if $Q\psi = 0$ then in the integral we can replace S' by any other $S' + Q\Omega$, with Ω of degree -1, and the integral is not affected, i.e.

$$\int \mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h\,\psi e^{iS'} = \int \mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h\,\psi e^{i(S'+Q\Omega)}.$$

Looking at (3) this means that we are allowed to replace the function $b^a f_a$ in

$$S' = S - Q(b^a f_a)$$

by some other function of degree -1 (for instance $b^a f'_a$ for some other set of functions f'_a) and the results remain unchanged. This is the BRST incarnation of the independence of the theory on the choice of gauge fixing.[3]

2.5. **Graded geometry.** Let us briefly summarise how the resulting "extended theory" looks like. First, our space of fields can be described as

$$\mathcal{M} := \mathbb{R}^n \times \mathfrak{g}[1] \times \mathfrak{g}[0] \times \mathfrak{g}[-1].$$

(We can call it the *BRST space* if we like.) What is this? It is a new version of "space", which is locally described by four sets of coordinates, x^i (taken to have degree 0), c^a (of degree 1), h^a (of degree 0), and b^a (of degree -1), where c^a, h^a, b^a correspond to (linear) coordinates on the vector space $\mathfrak{g}$ associated to some basis E_a.[4] On this "space" there are "functions" $C^\infty(\mathcal{M})$ (functionals of fields), which we obtain by multiplying the coordinates in various ways and using the rule that

$$y^\alpha y^\beta = (-1)^{\deg y^\alpha \deg y^\beta} y^\beta y^\alpha,$$

where y^α denotes collectively all coordinates on $\mathcal{M}$. The degree of a "function" is the sum of the degrees of its constituents. Lots of usual differential geometry carries over to this setup, by pretending that $\mathcal{M}$ is a real space and the above "functions" are really functions. One only needs to keep track of the fact that whenever we exchange two objects (functions, forms, vector fields, operators, etc.), we pick a sign just as in the above formula (this is called the *Koszul sign rule*).

For instance, as we saw above, $\mathcal{M}$ comes equipped with the degree 1 vector field

$$Q = c^a e^i_a(x)\partial_{x^i} - \tfrac{1}{2} f^a{}_{bc} c^b c^c \partial_{c^a} - h^a \partial_{b^a},$$

[3] You may wonder what would happen if, following this philosophy, we would add the exact term $Q(b^a f_a)$ to $S' = S - Q(b^a f_a)$, so that we would end up with the good old S. Well, we would immediately get $\int \mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h\,\psi e^{iS} = 0$ on account of the fermionic integrals. Why isn't this in conflict with the above independence of the theory on the choice of the gauge fixing? The answer is that one should in principle also keep track of the correct prefactor, instead of just writing $\sim$. When taking this nasty gauge fixing, the prefactor jumps to infinity, so that the only thing we can conclude is that taking $S' + Q(b^a f_a)$ is not a good idea.

[4] In general, for any vector space V, we denote by $V[n]$ the "space" with coordinates E^a (associated to a basis E_a of V) having degree n.

satisfying $Q^2 = 0$. (Such a graded space equipped with a degree 1 vector field Q squaring to zero is called a *dg manifold*.) $\mathcal{M}$ also carries a Q-invariant measure

$$\mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h.$$

It will however be more convenient to add also the new part of the exponential to the measure, i.e. we will instead consider the Q-invariant measure

$$\mu := \mathcal{D}x\,\mathcal{D}b\,\mathcal{D}c\,\mathcal{D}h\,e^{i(bAc+hf)}.$$

2.6. **What has happened?** Let us now step back and see what this was all about. We started with an (ordinary) manifold M with an action of G and an invariant measure μ_0 (in our case we had $M = \mathbb{R}^n$ and $\mu_0 = \mathcal{D}x$).

We then found a dg manifold $\mathcal{M}$, equipped with a map $\pi\colon \mathcal{M} \to M$ and a Q-invariant measure μ, such that G-invariant functions on M can be identified with $H^0(\mathcal{M})$ and for any G-invariant function Φ on M we have

$$\int_M \Phi\mu_0 = \int_{\mathcal{M}} (\pi^*\Phi)\mu.$$

In our case we were then interested in calculating this integral for $\Phi = \psi e^{iS}$. The point was that on the RHS the critical points (hopefully) become nondegenerate and we can employ the usual perturbative approach.

Remark. The setup discussed here can be used to deal with the simplest form of gauge symmetry, namely one that is not reducible. For instance, if we have a theory whose fundamental field is a p-form B on some Riemannian manifold N, with action $\int_N dB \wedge *dB$, then this has a natural gauge symmetry $B \mapsto B + d\Lambda$. This is however further reducible, i.e. Λ and $\Lambda + d\Omega$ lead to the same gauge transformation. So there are further gauge transformations for gauge transformations, etc. This can still be dealt with in the BRST framework, if in addition to ghosts we introduce the so-called ghosts-for-ghosts, etc. The framework is also powerful enough to deal with situations with field-dependent symmetry algebra.[5] There however exist also theories with a more complicated symmetry structure, for instance when the symmetry algebra only closes on-shell — these require a further generalisation of the BRST framework, which we now turn to. (We will see that it also sheds some new light onto the BRST formalism itself.)

3. BV formalism

3.1. **Symplectic geometry.** Recall that for any manifold N we can construct the cotangent bundle T^*N, which comes equipped with a symplectic form. Choosing a local coordinate system q^i on N, the cotangent bundle is locally described by coordinates q^i, p_i (i.e. to every q^i we assign a “dual” coordinate p_i), and the symplectic form is $\omega = dp_i \wedge dq^i$.

Note that any vector field $V \in \mathfrak{X}(N)$ can be seen as a linear function $\hat{V}$ on T^*N:

$$\mathfrak{X}(N) \ni V = V^i(q)\partial_{q^i} \quad\mapsto\quad \hat{V} := V^i(q)p_i \in C^\infty(T^*N).$$

One easily checks that this maps sends $[\,\cdot\,,\,\cdot\,]$ to $\{\,\cdot\,,\,\cdot\,\}$. Similarly, any $f \in C^\infty(N)$ can trivially be seen as a function (let's call it $\hat{f}$) on T^*N, and we have $\{\hat{V}, \hat{f}\} = \widehat{Vf}$.

A submanifold L of a symplectic manifold N is called *Lagrangian* if $\dim L = \frac{1}{2}\dim N$ and $\omega|_L = 0$. In particular (the image of) a section α of T^*N (which is the same as a 1-form on N) is Lagrangian if and only if $d\alpha = 0$. In particular any function $f \in C^\infty(N)$ gives rise to a Lagrangian submanifold of T^*N, corresponding to df.

[5] In other words, taking a commutator of two gauge transformations results in a linear combination of gauge transformations, which however have field-dependent coefficients.

More generally, we can play the same game if we start with a graded manifold $\mathcal{N}$ and then construct $T^*[k]\mathcal{N}$ as the space where to any coordinate q^i we assign a dual coordinate p^i such that $\deg q^i + \deg p_i = k$. This again carries a natural symplectic form $\omega = dp_i \wedge dq^i$ of degree k.[6] In what follows, we will consider the case $T^*[-1]\mathcal{N}$.

3.2. **Going to BV.** Let us now look back at the BRST construction. We will temporarily forget about the path integral itself and only look at the space of fields together with the classical dynamics and symmetries.

To start, recall that in the BRST story we ended up with a dg manifold $(\mathcal{M}, Q)$ together with a Q-invariant function S.[7] We now introduce a new larger space

$$\mathcal{M}_{BV} := T^*[-1]\mathcal{M}.$$

In order to simplify the notation we will denote the dual coordinates by an asterisk, so that $\mathcal{M}_{BV}$ is described by y^α and y^*_α. We also define a function

$$S_{BV} = \hat{S} + \hat{Q}$$

or more explicitly

$$S_{BV}(y, y^*) = S(y) + Q^\alpha(y)\, y^*_\alpha.$$

From the analysis in the previous subsection it follows that this automatically satisfies the *classical master equation*

$$\{S_{BV}, S_{BV}\} = 0$$

as a consequence of $QS = 0$ and $[Q, Q] = 2Q^2 = 0$.[8] The classical master equation is equivalent to saying that

$$Q_{BV} := \{S_{BV}, \,\cdot\,\}$$

is in fact a differential, turning $\mathcal{M}_{BV}$ into a dg symplectic manifold (i.e. a dg manifold with a symplectic form which is preserved by the differential).

The philosophy of the BV approach is however slightly different from BRST — one thing is that although we enlarged the space of fields yet again, we will not require the path integral to run over the entire $\mathcal{M}_{BV}$. Instead, one is supposed to perform the integral only over a Lagrangian submanifold of this big space. But which one?

Let Ψ be a function of degree -1 on $\mathcal{M}$, generating a Lagrangian submanifold $\mathcal{L} \subset T^*[-1]\mathcal{M} = \mathcal{M}_{BV}$. Recall that this is given by the condition

$$y^*_\alpha = \partial_{y^\alpha}\Psi(y).$$

We then have

$$S_{BV}|_{\mathcal{L}} = S(y) + Q^\alpha(y)\partial_{y^\alpha}\Psi(y) = S(y) + Q\Psi(y).$$

This is precisely what we obtained in the BRST approach (e.g. with $\Psi = -b^a f_a(x)$)! The extra terms in the BRST action (cf. (2)) now appear due to performing the path integral over a Lagrangian submanifold generated by Ψ rather than over the zero section of $T^*[-1]\mathcal{M}$ (which recovers simply S itself). Finally, we know from Observation 2 that the results of the physical calculations are independent of the

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[6] Note that here by "degree" we do not mean the differential form degree (symplectic form is always by definition a differential 2-form) but rather the total degree of all the coordinates which enter in ω, i.e. $\deg \omega = \deg p_i + \deg q^i = k$.

[7] What we here call S previously corresponded to $\pi^* S$. We apologise for the inconvenience.

[8] In graded geometry the commutator of vector fields U and V is defined as $[U, V] := UV - (-1)^{\deg U \deg V} VU$.

choice of Ψ — which in the BV framework correspond to invariance under the deformations of the Lagrangian submanifold in $\mathcal{M}_{BV}$.[9]

3.3. **Why is this helpful?** The general (classical) BV framework corresponds to taking an arbitrary graded symplectic manifold with ω of degree -1, and a degree 0 function S_{BV} satisfying the classical master equation. Such a framework can be used to capture a much more general class of theories than the ones admitting a BRST description. The latter embeds into the former via

$$(\mathcal{M}, Q, S) \quad \rightsquigarrow \quad (T^*[-1]\mathcal{M},\ S_{BV} = \hat{S} + \hat{Q}).$$

For instance the Faddeev–Popov procedure produces for us the symplectic space with

$$\omega = dx_i^* \wedge dx^i + dc_a^* \wedge dc^a + db_a^* \wedge db^a + dh_a^* \wedge dh^a,$$
$$S_{BV} = S(x) + c^a e_a^i(x) x_i^* - \tfrac{1}{2} f^a{}_{bc} c^b c^c c_a^* - h^a b_a^*,$$

as well as a Lagrangian submanifold generated by the function $-b^a f_a(x)$. Note that this BV space is a product of two BV spaces with

$$(\omega' = dx_i^* \wedge dx^i + dc_a^* \wedge dc^a, \quad S'_{BV} = S(x) + c^a e_a^i(x) x_i^* - \tfrac{1}{2} f^a{}_{bc} c^b c^c c_a^*)$$
$$(\omega'' = db_a^* \wedge db^a + dh_a^* \wedge dh^a, \quad S''_{BV} = -h^a b_a^*).$$

Notice that the second space is quite trivial — all the interesting stuff happens in the first one.

In contrast, a situation with $T^*[-1]\mathcal{M}$ but with S_{BV} having quadratic or higher terms in the dual variables y^* describes theories with symmetries that only close on-shell, and hence in particular do not admit a BRST description.

[9]In order to fully implement this independence in the path integral one needs the BV action to satisfy an $\hbar$-deformed version of the classical master equation, the so-called *quantum master equation*. We won't go that far in the present text.