A (VERY) SHORT COURSE IN CONFORMAL FIELD THEORY

FRIDRICH VALACH

1. Very brief introduction

This is a set of brief lecture notes for a very short course on conformal field theory delivered at MAPSS 2025 in Les Diablerets. In their preparation I have used the following sources:

- *Introduction to Conformal Field Theory* by Blumenhagen–Plauschinn
- *Lectures on String theory* by David Tong
- *Conformal Field Theory* by Di Francesco–Mathieu–Sénéchal
- *String theory, Volume 1: An Introduction to the Bosonic String* by Joseph Polchinski
- the text *Highest weight representations of the Virasoro algebra* from the master thesis of Jonas T. Hartwig (jthartwig.net/notes/virasoro.pdf)

I am also indebted to Samuel Valach for consultations.

2. Conformal group and algebra

Definition. *Consider any smooth manifold with a Riemannian metric g. A* conformal transformation *is a diffeomorphism under which the metric transforms as $g \to e^{\Lambda} g$ for some function $\Lambda(x)$.* Conformal group *is the group of all such transformations. The infinitesimal counterpart is given by the so-called* conformal Killing vector fields *— these are vector fields $V(x) = v^\mu(x)\partial_\mu$, preserving the metric up to overall scale, i.e.*

$$\partial_\mu v_\nu + \partial_\nu v_\mu = \lambda g_{\mu\nu}, \tag{1}$$

for some function $\lambda(x)$. They assemble into a conformal Lie algebra.

Proposition. *Let $M = \mathbb{R}^n$ for $n \geq 3$ with Euclidean metric. Then the conformal Lie algebra is isomorphic to $\mathfrak{so}(n+1,1)$.*

Let us now restrict our attention to the case $M = \mathbb{R}^2$, with Euclidean metric $g = (dx^0)^2 + (dx^1)^2$. Making a complex change of coordinates

$$z = x^0 + ix^1, \qquad \bar{z} = x^0 - ix^1,$$

the metric becomes $g = dz\, d\bar{z}$. For simplicity we define

$$\partial := \partial_z, \qquad \bar{\partial} := \partial_{\bar{z}},$$

and we rename the components of the vector field to $V = v^z\partial_z + v^{\bar{z}}\partial_{\bar{z}} =: v\partial + \bar{v}\bar{\partial}$. The equations (1) then reduce to

$$\bar{\partial} v = 0. \tag{2}$$

Since V is real, we have that $\bar{v}$ is in fact the complex conjugate of v, and thus the second equation $\partial\bar{v} = 0$ follows automatically. Equation (2) implies that the conformal Killing vector fields can be identified with holomorphic functions, and so the conformal Lie algebra is infinite-dimensional.

This analysis also holds if one takes an open subset of $\mathbb{C} \cong \mathbb{R}^2$ — the conformal Killing vector fields can be identified with holomorphic functions. A particularly interesting case is the one of punctured plane $M = \mathbb{C}^* \cong \mathbb{R}^2\backslash\{0\}$, in which case the basis of Killing vector fields is

$$l_n := -z^{n+1}\partial, \qquad n \in \mathbb{Z}.$$

Exercise. Show that $[l_m, l_n] = (m-n)l_{m+n}$. This is known as the *Witt algebra*. ◦

Even more generally, we can take M to be any 1-dimensional complex manifold. A particularly interesting example is the Riemann sphere $S^2 \cong \mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$.

Exercise. Show that on out of all l_n only l_{-1}, l_0, l_1 are globally well-defined on the Riemann sphere. (Hint: study the regularity of the vector field $v^z\partial_z$ at $z = \infty$, use the coordinate $w = 1/z$.) ◦

Exercise. Show that the subalgebra generated by l_{-1}, l_0, l_1 is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. ◦

Theorem 1. *The conformal group of the Riemann sphere is $SL(2,\mathbb{C})/\mathbb{Z}_2$, with the action*

$$SL(2,\mathbb{C}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}.$$

Remark. Note that when studying 2-dimensional CFT one typically treats z and $\bar{z}$ as independent complex variables, which effectively means that we embed $\mathbb{R}^2$ into $\mathbb{C}^2$. This allows us to use various tricks from complex analysis. Still, we must remember that in the end we are sitting at the subspace $\mathbb{R}^2 \subset \mathbb{C}^2$ where z and $\bar{z}$ are complex conjugates.

3. QFT and CFT

3.1. **Basics.** Ingredients in (Euclidean) *quantum field theory (QFT) with symmetry group G*:

- spacetime $M := \mathbb{R}^n$ with Euclidean metric $g = (dx^0)^2 + \cdots + (dx^{n-1})^2$
- complex Hilbert space $\mathcal{H}$, containing the vacuum state $|\text{vac}\rangle \in \mathcal{H}$
- *fields* = some chosen class of linear operators $\mathcal{O}_\alpha(x)$ on $\mathcal{H}$, depending on $x \in M$, satisfying *locality*:[1]
$$\mathcal{O}_\alpha(x)\mathcal{O}_\beta(y) = \mathcal{O}_\beta(y)\mathcal{O}_\alpha(x) \quad \text{if } x \neq y$$
and including the identity operator on $\mathcal{H}$
- projective[2] representation of G on $\mathcal{H}$, i.e. homomorphism $G \to GL(\mathcal{H})/\mathbb{C}^*$ preserving (the 1-dimensional subspace spanned by) $|\text{vac}\rangle$; the group G should contain at least the group of isometries of M.

It will be more convenient for us to trade the projective representation for an ordinary linear representation. This can be done, but in general one needs to pass from the group G to a central extension $\hat{G}$; this will then be represented on $\mathcal{H}$ linearly, i.e. via a homomorphism (for more details see the Appendix)

$$\hat{G} \to GL(\mathcal{H}).$$

Definition. *A* conformal field theory *is a QFT with G containing the conformal group of M.*

Basic objects to calculate in a QFT are the *correlators*

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \rangle := \langle \text{vac}| \, \mathcal{O}_1(x_1) \dots \mathcal{O}_k(x_k) \, |\text{vac}\rangle \in \mathbb{C},$$

where $\langle \rho | \, A \, |\phi\rangle :=$ inner product of $|\rho\rangle$ and $A\,|\phi\rangle$ (the *bra-ket notation*). The correlators can often (especially in CFT's) be effectively calculated using the *operator product expansion* (OPE)

$$\mathcal{O}_\alpha(x)\mathcal{O}_\beta(y) = \sum_\gamma f_{\alpha\beta\gamma}(x-y)\mathcal{O}_\gamma(y),$$

where f's are (hopefully) manageable ordinary functions, and the fact that the *1-point function* $\langle \mathcal{O}_\alpha(x) \rangle$ is nonzero iff $\mathcal{O}_\alpha$ is the identity operator.

3.2. **Examples of CFT.** Let us start by mentioning the important fact that many statistical models, such as the *Ising model*, *Potts model*, *$O(N)$-model*, etc., exhibit conformal symmetry near their critical points, and hence can be described by CFTs. We will return to these briefly later on.

Here we will instead focus on another class of models, related instead to string theory, where conformal invariance arises in a simple geometric way. To do that, suppose we start with a *classical field theory*, which contains the metric as one of the fundamental fields, i.e. the fundamental fields are $\{g_{\mu\nu}, \varphi_i\}$, and it is governed by an action $S(g,\varphi)$, which is invariant both under diffeomorphisms and the *Weyl transformations* $(g,\varphi) \mapsto (e^\omega g, \varphi)$, where $\omega \in C^\infty(M)$. Starting from such a setup, we claim that by fixing the metric variable g to some value $\hat{g}$ we obtain a new classical field theory

$$\hat{S}(\varphi) := S(\hat{g}, \varphi),$$

which is (classically) conformally invariant. Indeed, if now f is a conformal transformation for the fixed metric $\hat{g}$ (i.e. $f^*\hat{g} = e^\Lambda \hat{g}$) then we get

$$\hat{S}(f^*\varphi) = S(\hat{g}, f^*\varphi) \overset{\text{Weyl}}{=} S(f^*\hat{g}, f^*\varphi) \overset{\text{diffeo}}{=} S(\hat{g}, \varphi) = \hat{S}(\varphi),$$

[1] in the usual Lorentzian case we require fields to commute if their arguments are spacelike separated; when passing to the Euclidean setup by the Wick rotation, all pairs non-coincident points are spacelike separated

[2] It is not vectors but rays in $\mathcal{H}$ that are physically meaningful, hence the need for a projective representation.

and hence the theory is conformally invariant. When everything goes well, one obtains a CFT by quantising this latter theory.

Exercise (Polyakov action). Let M be a n-dimensional manifold, and (N, G) any Riemannian manifold. Consider the theory where the fields are metric g on M and a smooth map $X\colon M \to N$, with the action

$$S(g, X) = \int_M \omega_g (X^*G)_{\mu\nu} g^{\mu\nu} = \int_M \sqrt{\det g} g^{\mu\nu} \partial_\mu X^A \partial_\nu X^B G_{AB}(X) d^n x,$$

where $\omega_g = \sqrt{\det g}\, d^n x$ is the metric volume form on M. This action is automatically invariant under $\mathrm{Diff}(M)$. Verify that it is Weyl invariant iff $n = 2$. If $M = \mathbb{R}^2$, g is the constant Euclidean metric, and $N = \mathbb{R}^D$ then the resulting conformally invariant theory is the collection of D *free bosons*

$$\hat{S}(X) = \int_{\mathbb{R}^2} (\partial_\mu X^i)(\partial^\mu X_i) d^2 x = \tfrac{1}{2} \int_{\mathbb{C}} \partial X^i \bar{\partial} X_i dz d\bar{z}.$$

Write the action of l_n on X^i. ◦

Exercise (Yang–Mills action). Let $M = \mathbb{R}^n$ and $\mathfrak{g}$ a Lie algebra with invariant pairing. We take our fields to be the metric g and a 1-form A with values in $\mathfrak{g}$, with the action

$$S(g, A) = \tfrac{1}{2} \int_M \langle F_{\mu\nu}, F^{\mu\nu} \rangle \sqrt{\det g}\, d^n x = \int_M \langle F \wedge *F \rangle.$$

As before, this is automatically diffeomorphism invariant. Check that it is Weyl invariant iff $n = 4$.

Notably, however, this is one of the cases where "all does not go well" and the theory upon quantisation ceases to be conformally invariant. One way to remedy this is to pass to a particular supersymmetric extension, i.e. the famous $N = 4$ super Yang–Mills theory in 4 dimensions, which is conformal even after quantisation. ◦

4. Virasoro algebra and its representations

Theorem 2. *The Witt algebra admits a unique 1-dimensional central extension, called the* Virasoro algebra Vir*. This is spanned by elements L_n, $n \in \mathbb{Z}$, and an element $\hat{c}$, where $[\hat{c}, \mathrm{Vir}] = 0$ and*

$$[L_m, L_n] = (m - n)L_{m+n} + \tfrac{\hat{c}}{12}(m^3 - m)\delta_{m+n,0}.$$

Remark. By the 2nd Whitehead lemma $\mathfrak{so}(n+1, 1)$ does not admit any nontrivial central extension.

The relevant representations of Vir are the ones arising from the following construction:

Definition. *Denote by* Vir_+ *the subalgebra of* Vir *generated by elements L_n, $n \geq 0$ and $\hat{c}$. Let $R_{h,c}$ be the 1-dimensional representation of* Vir_+ *on which L_n, $n > 0$ acts trivially, L_0 acts by multiplication by $h \in \mathbb{R}$ and $\hat{c}$ by multiplication by $c \in \mathbb{R}$. We define the* Verma module

$$V(h, c) := \mathfrak{U}(\mathrm{Vir}) \otimes_{\mathfrak{U}(\mathrm{Vir}_+)} R_{h,c}.$$

Equivalently, define $V(h, c)$ as the infinite-dimensional representation of Vir*, whose basis is given by (the formal expressions)*

$$L_{-n_1} \dots L_{-n_k} |0\rangle, \qquad 0 < n_1 \leq \dots \leq n_k, \tag{3}$$

for $k = 0, 1, 2, \dots$ (in particular we have the basis element $|0\rangle$), the action of Vir *on $|0\rangle$ is given by the conditions*

$$L_0 |0\rangle = h |0\rangle, \qquad \hat{c} |0\rangle = c |0\rangle, \qquad L_n |0\rangle = 0 \text{ if } n > 0,$$

and the action of any L_n on any basis element (3) is determined by using these conditions together with the commutation relations of Vir

$$[L_m, L_n] = (m - n)L_{m+n} + \tfrac{\hat{c}}{12}(m^3 - m)\delta_{m+n,0}.$$

until one obtains a linear combination of expressions of the form (3).

For instance

$$\begin{aligned} L_2(L_{-1}L_{-2}|0\rangle) &= (L_2L_{-1})L_{-2}|0\rangle = (L_{-1}L_2 + [L_2, L_{-1}])L_{-2}|0\rangle = L_{-1}L_2L_{-2}|0\rangle + 3L_1L_{-2}|0\rangle \\ &= L_{-1}(L_{-2}L_2 + 4L_0 + \tfrac{c}{2})|0\rangle + 3(L_{-2}L_1 + 3L_{-1})|0\rangle \\ &= L_{-1}(4h + \tfrac{c}{2})|0\rangle + 9L_{-1}|0\rangle = (9 + 4h + \tfrac{c}{2})L_{-1}|0\rangle. \end{aligned}$$

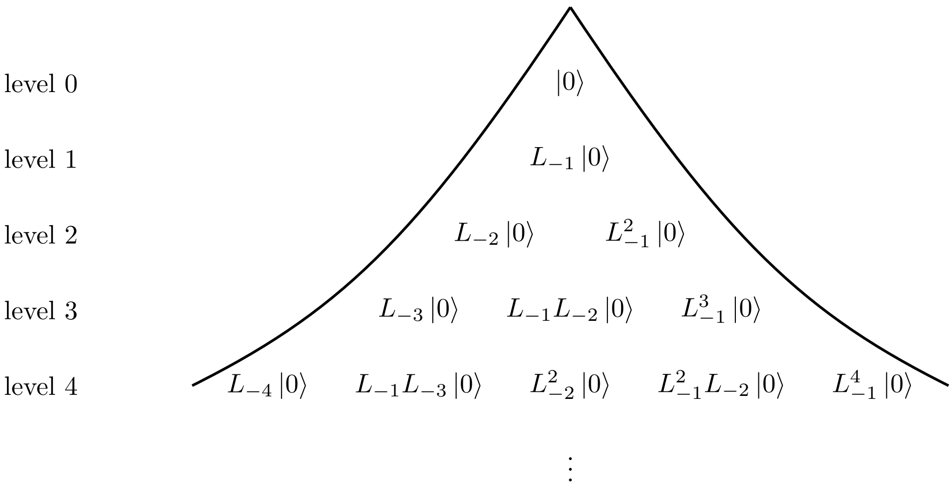
Definition. *The number c is called the* central charge. *The vector $|0\rangle \in V(h,c)$ is called the* primary state. *The other vectors (3) and their linear combinations are called* descendants. *The eigenvalues of L_0 are called* conformal dimensions. *Conformal dimension of a vector minus h is called the* level.

Exercise. Show that the action of L_n shifts the level by $-n$. In particular the level of the basis vector (3) is $n_1 + \cdots + n_k$. ◦

Thus the Verma module decomposes as a vector space into

$$V(h,c) = \bigoplus_{l=0}^{\infty} V_l(h,c),$$

where $V_l(h,c)$ consists of vectors of level l. The picture to have in mind is that of the "pyramid"



Exercise. Show

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{m=0}^{\infty} (\dim V_m(h,c)) q^m.$$ ◦

Definition. *We define a (unique) Hermitian form on the Verma module $V(h,c)$ by the following conditions:*

- *the pairing of $|0\rangle$ with itself (recall that this is denoted $\langle 0|0\rangle$) is defined to be* 1
- $L_n^\dagger = L_{-n}$.

For instance, the inner product of $L_{-2}|0\rangle$ with itself is

$$\langle 0| L_2 L_{-2} |0\rangle = \langle 0| (L_{-2}L_2 + 4L_0 + \tfrac{c}{2}) |0\rangle = 4h + \tfrac{c}{2}.$$

Note that $\langle 0| L_n = 0$ for $n < 0$.

Exercise. Calculate the inner product of $L_{-1}L_{-2}|0\rangle$ with itself, i.e. $\langle 0| L_2 L_1 L_{-1} L_{-2} |0\rangle$. ◦

Exercise. Show that if $k \neq l$ then $V_k(h,c) \perp V_l(h,c)$, i.e. different levels are mutually orthogonal. ◦

Definition. *A vector $|v\rangle$ is called* singular *if $L_n|v\rangle = 0$ for all $n > 0$ and* level $|v\rangle > 0$.[3]

Proposition. *Any singular vector v generates a proper subrepresentation* $\mathrm{Vir}\,|v\rangle \subset V(h,c)$*, which is orthogonal to the entire $V(h,c)$.*

Proof. First, note that the basis of the subrepresentation is given by

$$L_{-n_1} \dots L_{-n_k} |v\rangle, \qquad 0 < n_1 \leq \cdots \leq n_k. \tag{4}$$

Since the level of these elements is larger than 0, the subrepresentation does not contain $|0\rangle$ and hence is proper. To see that $\mathrm{Vir}\,|v\rangle \perp \mathrm{Vir}$, we calculate the inner product of elements of (3) and (4), which reduces to expressions of the form

$$\langle v| L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_k} |0\rangle, \qquad n_i, m_j > 0.$$

[3] In particular we suppose that v is an eigenvector of L_0.

Commuting the L_{n_i}'s to the right we either obtain $\dots L_{n_i}|0\rangle = 0$ or an expression of the above type with a smaller amount of L's. We perform this iteratively until there are no more $L_{>0}$ left, in which case we either have $\langle v| L_{<0} \cdots = 0$ or $\langle v|0\rangle$, which vanishes since $|v\rangle$ and $|0\rangle$ have different levels. □

Corollary. *The quotient $V(h,c)/\operatorname{Vir}|v\rangle$ inherits a well-defined Hermitian form.*

Proposition. *By quotienting out $V(h,c)$ by all subrepresentations generated by singular vectors one obtains an irreducible representation, which we will denote $I(h,c)$.*

We will really be interested in the representations $I(h,c)$. Note that for generic values of h and c there are no singular vectors in the Verma module and hence $V(h,c) = I(h,c)$.

Definition. *A representation $I(h,c)$ is called* unitary *if its Hermitian form is positive definite.*

It follows from the preceding discussion that if $V(h,c)$ contains a state of negative norm (often called a *ghost*) then $I(h,c)$ is not unitary.

Exercise. By calculating the norm of $L_{-n}|0\rangle$ show that $I(h,c)$ can only be unitary if $h \geq 0$ and $c \geq 0$. ∘

Theorem 3 (Friedan–Qiu–Shenker–Goddard–Kent–Olive)**.** *The irreducible highest weight representation $I(h,c)$ is unitary iff either*

- *$c \geq 1$ and $h \geq 0$, or*
- *$c = 1 - \frac{6}{m(m+1)} < 1$ for some $m \in \{2,3,4,\dots\}$ and $h = \frac{((m+1)r-ms)^2-1}{4m(m+1)}$ for some $r \in \{1,2,\dots,m-1\}$ and $s \in \{1,2,\dots,r\}$.*

5. Structure of 2-dimensional CFT

From now on we will exclusively study the 2-dimensional Euclidean setup, in complexified coordinates on $\mathbb{C}^*$. In order to be able to continue, let us mention (without discussing their origin in too much detail) some simplifying features of 2-dimensional CFTs.

Feature. *There exists a 1-1 map between fields and states of the Hilbert space, with the correspondence on $\mathbb{C}^*$ given by $\mathcal{O} \mapsto \mathcal{O}(0)|0\rangle$. The fields corresponding to primary states and descendants are called* primary fields *(or just* primaries*) and* descendants*. We will denote general fields by $\mathcal{O}_\alpha(z)$ and the primaries by Latin indices, e.g. $\mathcal{O}_i(z)$, with the associated conformal dimension h_i.*

We will effectively take this to be the definition of our class $\mathcal{O}_\alpha$ of fields (cf. QFT ingredients).

Feature. *In CFT in 2D it is usual for the holomorphic and antiholomorphic parts to "separate", i.e. the Hilbert space decomposes as (a sum of the blocks of the form)*

$$\mathcal{H}_{hol} \otimes \mathcal{H}_{antihol},$$

and each of the factors is acted upon by a separate copy of Vir *(so that overall we have* $\operatorname{Vir} \times \operatorname{Vir}$*). Furthermore, in a physically relevant theory the holomorphic (and similarly the antiholomorphic) part is given by*

$$\mathcal{H}_{hol} = \bigoplus_{h \in S} I(h,c)$$

for some fixed $c \in \mathbb{R}$ and some set $S \subset \mathbb{R}$.

We will assume this and from now focus only on the holomorphic (often also called *chiral*) part. Although this is not neccessary for the physics,[4] we will also require that I's are unitary.

Recall that the goal of the theory is to calculate the correlators and this can be achieved by knowing the OPE

$$\mathcal{O}_\alpha(z)\mathcal{O}_\beta(w) = \sum_\gamma f_{\alpha\beta\gamma}(z-w)\mathcal{O}_\gamma(w).$$

It can be shown that the condition of conformal invariance puts a heavy restriction on the possible form of the f's. For instance, it suffices to know the coefficients f_{ijk} for the primaries — the

[4]Many statistical models, such as the Yang–Lee edge singularity or some models of polymers admit a description in terms of non-unitary CFTs.

general $f_{\alpha\beta\gamma}$ can then be determined in a concrete algorithmic way independent of the particular CFT in question. Furthermore, one can show that

$$f_{ijk}(z) = c_{ijk} z^{h_k - h_i - h_j},$$

for some *structure constants* c_{ijk}. The theory is thus fully described by the following data:

- the central charge c
- the conformal weights h_i of primary operators
- the structure coefficients c_{ijk}.

This data however cannot be completely arbitrary and is further severely constrained (again by conformal invariance), so that often it suffices to specify only the central charge and the conformal weights and the rest can be uniquely determined. The program of determining consistent CFT's by studying these constraints is called *conformal bootstrap*.

Definition. *A CFT with S finite is called a* minimal model*.*

Proposition. *Unitary minimal models are classified by $m \in \{2, 3, \dots\}$. The central charge is then $c = 1 - \frac{6}{m(m+1)}$ and S is given by the collections of all allowed values of h from Theorem 3.*

One enormous success in the field of CFT has been the identification of various statistical models (at their critical points) with minimal models, and their subsequent solution. For instance:

- for $m = 2$ we have $c = 0$ and $S = \{0\}$: this corresponds to the "empty theory" with only identity operator
- for $m = 3$ we have $c = 1/2$ and $S = \{0, 1/16, 1/2\}$: this is the critical Ising model in 2d, the fields with weights 1/16 and 1/2 correspond to spin and energy density, respectively
- for $m = 4$ we have $c = 7/10$ and $S = \{0, 3/80, 1/10, 7/16, 3/5, 3/2\}$: the tricritical Ising model
- for $m = 5$ we have $c = 4/5$ and $S = \dots$: three-state Potts model
- ...

Often the model has a larger symmetry, which extends Virasoro — a notable example is the Wess–Zumino–Witten model, where the symmetry involves an affine Lie algebra. CFTs whose Hilbert space contains finitely many representations of this enlarged symmetry algebra are called *rational*.

6. Operator product expansion and the energy-momentum tensor

It turns out that in every CFT there is an important operator, linked to the conformal transformations, called the *energy-momentum tensor* (also known as the *stress-energy tensor* or *stress-energy-momentum tensor*). Focusing again on the holomorphic (chiral) part, this is given by a holomorphic operator $T(z)$,[5] which is related to the Virasoro generators by the Laurent expansion

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-n-2} L_n.$$

The connection between this operator and conformal transformations then translates into the following OPE statements.

Before starting, let us note that the operators we will be interested in have the OPE of the form

$$\mathcal{O}_1(z)\mathcal{O}_2(w) = \sum_{n=-\infty}^{N} \frac{\mathcal{O}_{\alpha_n}(w)}{(z-w)^n}. \tag{5}$$

Since we will mostly study the singular part, we will use the notation where $\sim$ denotes equality up to regular terms, i.e.

$$\mathcal{O}_1(z)\mathcal{O}_2(w) \sim \sum_{n=1}^{N} \frac{\mathcal{O}_{\alpha_n}(w)}{(z-w)^n}.$$

Now one can show that an operator $\mathcal{O}$ is primary (of weight h) iff its OPE with T has the form

$$T(z)\mathcal{O}(w) \sim h\frac{\mathcal{O}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}(w)}{z-w}. \tag{6}$$

[5]See the Appendix for more details on the definition, geometry, and properties of the energy-momentum tensor.

Furthermore, one can show that the OPE of T (in a theory with central charge c) with itself is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{7}$$

Operator T is thus not primary, unless $c = 0$.

Definition. *Suppose we have an OPE (5) of two fields. We define the* contraction *as the singular part, i.e.*

$$\overline{\mathcal{O}_1(z)\mathcal{O}_2}(w) := \sum_{n=1}^{N} \frac{\mathcal{O}_{\alpha_n}(w)}{(z-w)^n}.$$

We also define the normal ordering *of $\mathcal{O}_1$ and $\mathcal{O}_2$ as the field denoted $(\mathcal{O}_1\mathcal{O}_2)(z)$ given by*

$$(\mathcal{O}_1\mathcal{O}_2)(w) := \mathcal{O}_{\alpha_0}(w) = \lim_{z\to w}[\mathcal{O}_1(z)\mathcal{O}_2(w) - \overline{\mathcal{O}_1(z)\mathcal{O}_2}(w)].$$

We can also conveniently write

$$(\mathcal{O}_1\mathcal{O}_2)(w) = \frac{1}{2\pi i}\oint_w \frac{dz}{z-w}\mathcal{O}_1(z)\mathcal{O}_2(w),$$

where $\oint_w$ is the integral along a (small) circle around w.

Proposition (Wick theorem)**.**

$$\overline{\mathcal{O}_1(z)(\mathcal{O}_2}\mathcal{O}_3)(w) = \frac{1}{2\pi i}\oint_w \frac{dx}{x-w}\left(\overline{\mathcal{O}_1(z)\mathcal{O}_2}(x)\mathcal{O}_3(w) + \mathcal{O}_2(x)\overline{\mathcal{O}_1(z)\mathcal{O}_3}(w)\right).$$

Proposition. $\partial(\mathcal{O}_1\mathcal{O}_2) = (\partial\mathcal{O}_1\mathcal{O}_2) + (\mathcal{O}_1\partial\mathcal{O}_2)$.

Proof. Making a change of variables $z \to z + \epsilon$ in the second step we get

$$\begin{aligned}
\partial(\mathcal{O}_1\mathcal{O}_2) &= \lim_{\epsilon\to 0}\frac{1}{2\pi i\epsilon}\left(\oint_{w+\epsilon}\frac{\mathcal{O}_1(z)\mathcal{O}_2(w+\epsilon)dz}{z-w-\epsilon} - \oint_w \frac{\mathcal{O}_1(z)\mathcal{O}_2(w)dz}{z-w}\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi i\epsilon}\oint_w \frac{\mathcal{O}_1(z+\epsilon)\mathcal{O}_2(w+\epsilon) - \mathcal{O}_1(z)\mathcal{O}_2(w)}{z-w}dz \\
&= \frac{1}{2\pi i}\oint_w \frac{\partial\mathcal{O}_1(z)\mathcal{O}_2(w) + \mathcal{O}_1(z)\partial\mathcal{O}_2(w)}{z-w}dz = (\partial\mathcal{O}_1\mathcal{O}_2) + (\mathcal{O}_1\partial\mathcal{O}_2).
\end{aligned}$$

□

Exercise. Find the definition of a *vertex operator algebra* and persuade yourself that this is a structure describing the chiral part of a CFT. ○

7. Example: free boson

Let us now develop in more detail the example of the free boson. The most important (holomorphic) primary field in the game is ∂X, which has the following OPE with itself[6]

$$\partial X(z)\partial X(w) \sim -\frac{1}{(z-w)^2} \tag{8}$$

The energy-momentum tensor is

$$T(z) = \gamma(\partial X\partial X)(z),$$

where γ is some constant (to be determined).[7] Let us now check that for a suitable γ we get both that ∂X is a primary field of some conformal weight h and that T is a correct energy-momentum tensor; we will also calculate the central charge.

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[6]This can be determined for instance from the propagator for the field X. Note that despite the fact that the corresponding classical field theory is formulated using the field X, this field is not the most convenient object to work with in the quantum version of the story. Instead, we use ∂X. Also note that the classical equations of motion say $\bar\partial(\partial X) = 0$ — and the same equation can also be used under the correlators, as long as there is no collision with other operator insertions (cf. the last remark of the first part of the Appendix).

[7]Again, this comes from looking at the classical energy-momentum tensor, whose holomorphic part is simply $T = \partial X\partial X$. This has to be regularised (and normalised) in the quantum version of the story, hence the normal ordering (which excludes the singular bits of the product of two operators at a point).

First, let us calculate

$$\partial X(z)T(w) \sim \gamma \overbrace{\partial X(z)(\partial X}\partial X)(w) = -\frac{\gamma}{2\pi i}\oint_w \frac{dx}{x-w}\left(\frac{1}{(z-x)^2}\partial X(w) + \frac{1}{(z-w)^2}\partial X(x)\right)$$
$$= -2\gamma\frac{\partial X(w)}{(z-w)^2},$$

where we used the fact that for a holomorphic function $f(z)$ we have

$$\frac{1}{2\pi i}\oint_w \frac{dz}{z-w} f(z) = f(w).$$

However, we actually want to have the result and arguments in the form (6). Hence we continue:

$$T(z)\partial X(w) \sim -2\gamma\frac{\partial X(z)}{(z-w)^2} \sim -2\gamma\frac{\partial X(w)}{(z-w)^2} - 2\gamma\frac{\partial(\partial X(w))}{z-w},$$

where we Taylor-expanded $\partial X(z)$ around the point w. Comparing the last term with the one in (6) we see that in order for ∂X to be primary we need to set $\gamma = -\frac{1}{2}$, and so

$$T(z) = -\tfrac{1}{2}(\partial X \partial X)(z), \qquad T(z)\partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w}.$$

In particular we see that ∂X is primary of weight 1.

In order to calculate the OPE of T with itself we first take a derivative of (8) w.r.t. z to get

$$\partial^2 X(z)\partial X(w) \sim \frac{2}{(z-w)^3}$$

and a similar result for derivative w.r.t. w. Using the OPE of T with ∂X we then obtain

$$T(z)T(w) \sim -\tfrac{1}{2}\overbrace{T(z)(\partial X}\partial X)(w)$$
$$= -\frac{1}{4\pi i}\oint_w \frac{dx}{x-w}\left[\left(\frac{\partial X(x)}{(z-x)^2} + \frac{\partial^2 X(x)}{z-x}\right)\partial X(w) + \left(\frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w}\right)\partial X(x)\right].$$

Looking at the first term, we note that substituting

$$\partial X(x)\partial X(w) = -\frac{1}{(x-w)^2} + (\partial X\partial X)(w) + \text{terms proportional to } (x-w)$$

the last bit, i.e. the terms proportional to $(x-w)$, does not actually contribute to the integral. Performing such an expansion for every term and using $T = -\frac{1}{2}(\partial X\partial X)$ and $\partial T = -(\partial^2 X \partial X)$ we finally get

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

showing that the free boson is a conformal field theory with central charge $c = 1$.

Exercise. Finish the last step in the above calculation. ◦

8. Other examples

The examples here are based on Polchinski's book String theory and the Conformal Field Theory book by Di Francesco–Mathieu–Sénéchal.

8.1. **Linear dilaton CFT.** We start as in free boson with the field ∂X with

$$\partial X(z)\partial X(w) \sim -\frac{1}{(z-w)^2}$$

but now with the energy-momentum tensor

$$T = -\tfrac{1}{2}(\partial X\partial X) + v\partial^2 X$$

for some $v \in \mathbb{R}$. Show that ∂X is no longer primary, but T still satisfies (7) with the central charge

$$c = 1 + 12v^2.$$

8.2. bc **system.** Since this is going to be a theory describing fermions, we will need to make a small adjustment to what was said before — the fields $b(z)$ and $c(w)$ for $z \neq w$ will now no longer commute but rather anticommute with each other,

$$b(z)c(w) = -c(w)b(z), \qquad b(z)b(w) = -b(w)b(z), \qquad c(z)c(w) = -c(w)c(z).$$

The rest of the discussion is unchanged.[8] The relevant OPEs are now

$$b(z)c(w) \sim \frac{1}{z-w}, \qquad b(z)b(w) \sim 0, \qquad c(z)c(w) \sim 0$$

and the energy-momentum tensor is

$$T = (\partial b\, c) - \lambda \partial(bc),$$

for some $\lambda \in \mathbb{R}$. Derive the "opposite" OPE $c(z)b(w) \sim \frac{1}{z-w}$ and show that for every λ this theory defines a CFT with the central charge

$$c = -3(2\lambda - 1)^2 + 1,$$

and with the conformal weights of the fields b, c being λ and $1-\lambda$, respectively. In the case $\lambda = 2$ (hence $c = -26$) this theory appears in the gauge-fixing of the diffeomorphism symmetry of the Polyakov action in string theory.

8.3. $\beta\gamma$ **system.** There is a bosonic (commuting) analogue of the bc system, called the $\beta\gamma$ system, where we have

$$\beta(z)\gamma(w) \sim -\frac{1}{z-w}, \qquad \beta(z)\beta(w) \sim 0, \qquad \gamma(z)\gamma(w) \sim 0,$$

and the energy-momentum tensor is

$$T = (\partial\beta\, \gamma) - \lambda \partial(\beta\gamma).$$

Calculate the OPE $\gamma(z)\beta(w) \sim \frac{1}{z-w}$ and show that this theory defines a CFT with the central charge opposite to the bc system, i.e.

$$c = 3(2\lambda - 1)^2 - 1,$$

and that the weights of β and γ are λ and $1-\lambda$, respectively. For $\lambda = \frac{3}{2}$ (hence $c = 11$) this theory appears in the gauge-fixing of the superstring.

8.4. **Wess–Zumino–Witten model.** Fix a compact simple Lie group G with Lie algebra $\mathfrak{g}$ and fix an integer k (this is called the *level*). Choose an arbitrary basis e_a of $\mathfrak{g}$ and denote the structure constants by ${c_{ab}}^c = [e_a, e_b]^c$.

Let now K be the *normalised Killing form*[9] on $\mathfrak{g}$, i.e.

$$K_{ab} = \frac{1}{2h^\vee} {c_{ac}}^d {c_{bd}}^c,$$

where $h^\vee$ is the *dual Coxeter number*.[10] (Notably for $\mathfrak{su}(n)$ the normalised Killing form coincides precisely with the matrix trace, $K(x,y) = \mathrm{tr}(xy)$.) The Killing form is a non-degenerate (in fact negative-definite) metric on $\mathfrak{g}$, and from now on we will use it (and its inverse, denoted as usual by K^{ab}) to lower and raise any indices if needed.

Exercise. Show that c_{abc} is totally antisymmetric. ∘

The basic ingredient of the (chiral part of the) WZW model is the field $J(z)$, valued in $\mathfrak{g}$. Equivalently, using the basis we have a set of fields $\{J^a(z)\}_{a=1}^{\dim \mathfrak{g}}$. We impose the following OPE:

$$J^a(z)J^b(w) \sim \frac{kK^{ab}}{(z-w)^2} + \frac{{c^{ab}}_c J^c(w)}{z-w}.$$

We define the energy-momentum tensor by

$$T = \gamma(J^a J_a),$$

[8] There is also a relative minus sign on the RHS of the Wick theorem, due to the swapping of $\mathcal{O}_1$ and $\mathcal{O}_2$ in the second term.

[9] It is normalised by the condition that the long roots of $\mathfrak{g}$ have length squared equal to 2. This won't be important for what follows.

[10] This is defined as $h^\vee = 1 + \sum n_i^\vee$, where $n_i^\vee$ are the expansions coefficients of the maximal coroot in the basis of simple coroots. Again, you will not need to know nor understand these notions for the present purposes.

where γ is a constant to be determined. This is called the *Sugawara construction.*[11]

Exercise. Fix γ by demanding

$$T(z)J^a(w) \sim h\frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}$$

for some h. (Answer: $\gamma = \frac{1}{2(k+h^\vee)}$ and $h = 1$.) ◦

Exercise. Using this value of γ calculate the OPE of T with itself and determine the central charge. (Answer: $c = \frac{k \dim \mathfrak{g}}{k+h^\vee}$.) ◦

Appendix: Energy-momentum tensor

Suppose now that we have an arbitrary classical field theory on a manifold M containing the metric, described by $S(g, \varphi)$.

Definition. *We define the* energy-momentum tensor *(also known as the* stress-energy tensor *or* stress-energy-momentum tensor*) using the variational derivative as follows:*

$$T^{\mu\nu} := \frac{1}{\sqrt{\det g}} \frac{\delta S}{\delta g_{\mu\nu}}.$$

In other words, upon an infinitesimal change of metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ the action changes as

$$S \to S + \delta S, \qquad \delta S = \int_M T^{\mu\nu} \delta g_{\mu\nu} \sqrt{\det g}\, d^n x = \int_M T^{\mu\nu} \delta g_{\mu\nu} \omega_g.$$

Note that the tensor T depends both on g and φ and is symmetric.

Remark. The energy-momentum tensor is to be thought as representing the density and flux of energy and momentum for the theory in question. For instance, when in general relativity one considers a system governed by the sum of the Einstein–Hilbert action and a matter action $S(g, \varphi)$, the corresponding equations of motion reproduce precisely the Einstein equations

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

together with the equations of motion for the matter fields φ, given by $\delta S(g, \varphi)/\delta \varphi_i = 0$.

Exercise. Show that the stress-energy tensor for the Polyakov action is

$$T_{\mu\nu} = \partial_\mu X^i \partial_\nu X_i - \tfrac{1}{2} g_{\mu\nu} \partial_\rho X^i \partial^\rho X_i.$$ ◦

Proposition. *Suppose S is diffeomorphism invariant. Then the energy-momentum tensor satisfies $\nabla_\mu T^{\mu\nu} = 0$ on-shell, i.e. at the critical locus of the action $\hat{S}$. Here ∇ is the Levi-Civita connection for $\hat{g}$.*

Proof. Take any vector field v on M and use it to deform $\varphi \to \varphi + \epsilon L_v \varphi$. Then

$$\begin{aligned} \delta \hat{S} &= \hat{S}(\varphi + \epsilon L_v \varphi) - \hat{S}(\varphi) = S(\hat{g}, \varphi + \epsilon L_v \varphi) - S(\hat{g}, \varphi) = S(\hat{g} - \epsilon L_v \hat{g}, \varphi) - S(\hat{g}, \varphi) \\ &= -\epsilon \int_M T^{\mu\nu} (L_v \hat{g})_{\mu\nu} \omega_{\hat{g}} = -2\epsilon \int_M T^{\mu\nu} (\nabla_\mu v_\nu) \omega_{\hat{g}} = 2\epsilon \int_M (\nabla_\mu T^{\mu\nu}) v_\nu \omega_{\hat{g}}, \end{aligned}$$

where we used $(L_v \hat{g})_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$. Since on-shell we have $\delta \hat{S} = 0$ for any v, it follows that at this locus $\nabla_\mu T^{\mu\nu} = 0$. □

Proposition. *If S is Weyl invariant then the energy-momentum tensor is traceless: $T^\mu{}_\mu = 0$.*

Proof. Under an infinitesimal Weyl transformation $g \to g + \epsilon \lambda g$ (for λ any function) we have

$$0 = S(g + \epsilon \lambda g, \varphi) - S(g, \varphi) = \epsilon \int_M T^{\mu\nu} \lambda g_{\mu\nu} \omega_g = \epsilon \int_M T^\mu{}_\mu \lambda \omega_g.$$ □

Exercise. Check that the energy-momentum tensor for the Polyakov action is traceless iff $n = 2$. ◦

[11] Recall that the Laurent coefficients of T are the generators of the Virasoro algebra. Similarly, the Laurent coefficients of J are the generators of the so-called *affine Lie algebra* (associated to $\mathfrak{g}$). The Sugawara construction then describes a concrete embedding of the Virasoro algebra into the universal envelopping algebra of this affine Lie algebra.

Corollary. *Let $S(g,\varphi)$ be diffeomorphism and Weyl-invariant, and $\hat{S}(\varphi) = S(\hat{g},\varphi)$ the corresponding conformally invariant theory. Then the conserved current associated to a conformal Killing vector field ζ is $T^{\mu\nu}\zeta_\nu$, i.e. on-shell (for $\hat{S}$) we have $\nabla_\mu(T^{\mu\nu}\zeta_\nu) = 0$.*

Proof. On-shell we have $\nabla_\mu(T^{\mu\nu}\zeta_\nu) = T^{\mu\nu}\nabla_\mu\zeta_\nu = \frac{1}{2}\lambda T^{\mu\nu}g_{\mu\nu} = 0$. □

Remark. One can also formally derive the form of this current directly using the *Noether method*: we consider a new vector field $f\zeta$, where f is a function on M and study the change of $\hat{S}$ under this transformation. As before, using the diffeomorphism invariance to trade off the deformation of φ for a deformation of g, we get

$$\begin{aligned}\delta\hat{S} = S(\hat{g} - L_{f\zeta}\hat{g},\varphi) - S(\hat{g},\varphi) &= -2\int_M T^{\mu\nu}\nabla_\mu(f\zeta_\nu)\omega_g = -2\int_M T^{\mu\nu}(f\nabla_\mu\zeta_\nu + \zeta_\nu\nabla_\mu f)\omega_g \\ &= -\int_M T^{\mu\nu}(f\lambda g_{\mu\nu} + 2\zeta_\nu\nabla_\mu f)\omega_g = 2\int_M \nabla_\mu(T^{\mu\nu}\zeta_\nu)f\omega_g.\end{aligned}$$

Since on-shell we have $\delta\hat{S}$ for any f, it follows that $\nabla_\mu(T^{\mu\nu}\zeta_\nu) = 0$.

Exercise. For any 2-dimensional CFT $\hat{S}$ (on a 1-dimensional complex manifold) arising from $S(g,\varphi)$ show that the only nonzero components of the energy-momentum tensor are

$$T := T_{zz}, \qquad \bar{T} := T_{\bar{z}\bar{z}}$$

and that on-shell (for $\hat{S}$) they satisfy

$$\bar{\partial}T = 0, \qquad \partial\bar{T} = 0.$$ ○

Claim. *One can easily show using a path-integral argument that the differential equations which classically hold on-shell still hold "under the correlators as long as the other field insertions happen at other points", for instance*

$$\langle\varphi(x_1)\partial\varphi(x_2)\bar{\partial}T(x)\rangle = 0$$

as long as $x \neq x_1$ and $x \neq x_2$. That's why (e.g. in OPEs) we often pretend that the operator T itself is holomorphic (while in fact this is true only under the correlators).

Appendix: Projective to linear

Suppose we start with a projective representation of a Lie group G, i.e. a (say complex) vector space V and a homomorphism

$$\rho\colon G \to GL(V)/\mathbb{C}^*. \tag{9}$$

We can ask: is it possible that there exists a linear representation, i.e. a homomorphism

$$\rho'\colon G \to GL(V)$$

which gives rise to the former projective representation after composing with

$$\pi\colon GL(V) \to GL(V)/\mathbb{C}^*,$$

i.e. $\rho = \pi \circ \rho'$? The general answer is "No". However, one can always make things work by enlarging the group slightly. In more detail:

Starting with a projective representation (9) we define a new group

$$\hat{G} \subset G \times GL(V), \qquad \hat{G} = \{(g,A) \mid \rho(g) = \pi(A)\}.$$

It is not difficult to see that this is a (1-dimensional) central extension of G, i.e. there is an exact sequence of groups

$$1 \to \mathbb{C}^* \to \hat{G} \to G \to 1, \tag{10}$$

s.t. the image of $\mathbb{C}^*$ is contained in the centre of $\hat{G}$. The first map in (10) is $\mathbb{C}^* \ni z \mapsto (1, z\,\mathrm{id}) \in \hat{G}$ and the second one is $\hat{G} \ni (g,A) \mapsto g \in G$. The new group $\hat{G}$ furthermore admits a linear representation on V, simply by $\hat{G} \ni (g, A) \mapsto A \in GL(V)$ and this is compatible with the projective representation of G in the sense that the following commutes

$$\begin{array}{ccc} \hat{G} & \longrightarrow & G \\ \downarrow & & \downarrow \rho \\ GL(V) & \xrightarrow{\pi} & GL(V)/\mathbb{C}^* \end{array}$$

Sometimes an even smaller central extension suffices. Notably, one has:

Theorem 4 (Bargmann)**.** *Suppose G is a Lie group whose Lie algebra has vanishing 2nd cohomology. Then every projective unitary representation of G can be made linear by passing to the universal cover of G.*

Note that taking a universal cover of a Lie group can also be regarded as a (0-dimensional) central extension (10) where $\mathbb{C}^*$ is replaced by a discrete abelian group. Importantly, the 2nd cohomology vanishes for any semisimple Lie algebra (this is the 2nd Whitehead lemma).

This is for instance how $SU(2)$ representations arise in quantum mechanics of half-integer spins, since one is interested in projective representations of the rotation group $SO(3) = SU(2)/\mathbb{Z}_2$.