Basics of generalised geometry

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Abstract: In these lectures we look at some basics of generalised geometry. We start with a brief recollection of some standard notions from differential geometry, and some examples of geometric structures. We then arrive at (examples of) Lie and Courant algebroids by looking at symmetries of diffeomorphism-invariant theories — Einstein–Yang–Mills and the universal sector of stringy supergravity, respectively. After that we dive shortly into some elementary theory of Courant algebroids and Dirac structures and show how they give a unified viewpoint of Poisson, symplectic, and complex geometry. We finish with application to string theory, namely to its low-energy dynamics and dualities.

Contents

1 Preliminaries

(Be warned that this is a work still under construction, and there are potentially mistakes/errors. Feel free to let me know if you find any of them. The plan is to turn this into a more readable set of lecture notes.)

Differential forms

- notation: $\Omega^p(M)$
- *wedge product* (*exterior product*): $\alpha \wedge \beta = (-1)^{\alpha\beta} \beta \wedge \alpha$

  $(\Omega^p, \wedge)$ is a graded commutative algebra
- an $\mathbb{R}$-linear operator D on $\Omega^p(M)$ is called a *derivation of degree* n if

$$D\Omega^p \subset \Omega^{p+n}, \qquad D(\alpha \wedge \beta) = (D\alpha) \wedge \beta + (-1)^{n \deg \alpha} \alpha \wedge D\beta$$

  $\deg D := n \quad \rightsquigarrow$ Koszul sign rule, $\mathrm{Der}_n :=$ the space of derivations of degree n

$$D \in \mathrm{Der}_n,\ D' \in \mathrm{Der}_{n'} \quad \Longrightarrow \quad [\![D, D']\!] := DD' - (-1)^{nn'} D'D \ \in \mathrm{Der}_{n+n'}$$

  ($\mathrm{Der}_\bullet$ is a graded Lie algebra)
- *Lie derivative* $\mathcal{L}_X \in \mathrm{Der}_0$ (for $X \in \Gamma(TM)$), i.e. $\mathcal{L}_X(\alpha \wedge \beta) = (\mathcal{L}_X \alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$
- *exterior derivative* $d \in \mathrm{Der}_1$, i.e. $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$
- *interior product* $i_X \in \mathrm{Der}_{-1}$, i.e. $i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X \beta$

$$\begin{aligned}
[\![\mathcal{L}_X, \mathcal{L}_Y]\!] &\equiv \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X = \mathcal{L}_{[X,Y]_{\mathfrak{X}}} \\
[\![\mathcal{L}_X, i_Y]\!] &\equiv \mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]_{\mathfrak{X}}} \\
[\![d, i_X]\!] &\equiv d i_X + i_X d = \mathcal{L}_X \\
[\![d, d]\!] &\equiv 2d^2 = 0 \\
[\![i_X, i_Y]\!] &\equiv i_X i_Y + i_Y i_X = 0 \\
[\![\mathcal{L}_X, d]\!] &\equiv \mathcal{L}_X d - d\,\mathcal{L}_X = 0
\end{aligned}$$

Symplectic and Poisson geometry

- *symplectic structure* on M is $\omega \in \Omega^2(M)$ such that $d\omega = 0$ and ω is nondegenerate

  nondegeneracy means $\omega^\flat \colon TM \to T^*M$ is an isomorphism (ω_{ij} is an invertible matrix)
- inverting $\omega^\flat$ (or ω_{ij}) we obtain an antisymmetric tensor $\pi \in \mathcal{T}^2_0(M)$ (bivector)

  define the Poisson bracket of functions on M by $\{f, g\} := \pi(df, dg)$

  $d\omega = 0 \implies$ Jacobi for $\{\,\cdot\,,\,\cdot\,\}$
- *Poisson structure* on M is a bivector field π whose induced Poisson bracket satisfies Jacobi
- symplectic $\implies$ Poisson, but not the other way round (e.g. $\pi = 0$ is a Poisson structure)
- for $H \in C^\infty(M)$ we have a vector field $X_H := \pi(dH,\,\cdot\,)$

$$X_H f = \langle \pi(dH,\,\cdot\,), df\rangle = \pi(dH, df) = \{H, f\}$$

- M serves as a phase space, X_H gives the time evolution of the system
- time evolution of *observables* $f \in C^\infty(M)$ is given by $\dot{f} = X_H f = \{H, f\}$
- example: $M = \mathbb{R}^{2n}$, $\omega = dp_i \wedge dq^i \rightsquigarrow \{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i}$, $\dot{q}^i = \frac{\partial H}{\partial p_i}$, $\dot{p}_i = -\frac{\partial H}{\partial q^i}$
- example: $M = S^2$, $\omega =$ volume form

Complex structure

- complex manifold = charts valued in $\mathbb{C}^n$, holomorphic transition functions
- coordinates z^μ give rise to real coordinates $z^\mu = x^\mu + iy^\mu$
- define tensor $J \in \mathcal{T}^1_1(M)$ by $J\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial y^\mu}$, $J\frac{\partial}{\partial y^\mu} = -\frac{\partial}{\partial x^\mu} \quad \rightsquigarrow \quad J^2 = -1$
- *almost complex structure* is a tensor $J \in \mathcal{T}^1_1(M)$ such that $J^2 = -1$
- a.c.s. is called *integrable* (or simply *complex structure*) if it corresponds to a complex manifold
- *Nijenhuis tensor* $N_J \in \mathcal{T}^1_2(M)$, $N_J(X, Y) := [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]$
- (Newlander–Nirenberg theorem) Almost complex structure is integrable iff $N_J = 0$.
- Exercise: given an a.c.s., take $L \subset TM \otimes \mathbb{C}$ to be the -1-eigenbundle of J; show that $N_J = 0$ iff $[\Gamma(L), \Gamma(L)]_{\mathfrak{X}\otimes\mathbb{C}} \subset \Gamma(L)$

2 Algebroids

Symmetries of the Einstein–Yang–Mills theory

$$S(g, A) = \int_M R\,\mathrm{vol}_g + F \wedge *F \qquad g \text{ metric, } A \in \Omega^1(M)$$

- abelian case: symmetries generated by $X \in \Gamma(TM)$, $\lambda \in \Omega^0(M)$

$$\delta_{(X,\lambda)} g = \mathcal{L}_X g, \qquad \delta_{(X,\lambda)} A = \mathcal{L}_X A + d\lambda$$

- calculating the algebra of symmetries $\delta_{(X,\lambda)}\delta_{(X',\lambda')} - \delta_{(X',\lambda')}\delta_{(X,\lambda)}$, looking at A:

$$
\begin{aligned}
&A \xrightarrow{(X',\lambda')} A + \epsilon(\mathcal{L}_{X'}A + d\lambda') \xrightarrow{(X,\lambda)} (A + \epsilon(\mathcal{L}_{X'}A + d\lambda')) + \epsilon(\mathcal{L}_X(A + \epsilon(\mathcal{L}_{X'}A + d\lambda')) + d\lambda) \\
&\quad = A + \epsilon(\mathcal{L}_X A + \mathcal{L}_{X'}A + d\lambda + d\lambda') + \epsilon^2(\mathcal{L}_X\mathcal{L}_{X'}A + \mathcal{L}_X d\lambda')
\end{aligned}
$$

$$
\begin{aligned}
(\delta_{(X,\lambda)}\delta_{(X',\lambda')} - \delta_{(X',\lambda')}\delta_{(X,\lambda)})A &= \mathcal{L}_X\mathcal{L}_{X'}A - \mathcal{L}_{X'}\mathcal{L}_X A + d\mathcal{L}_X\lambda' - d\mathcal{L}_{X'}\lambda \\
&= \mathcal{L}_{[X,X']_{\mathfrak{X}}}A + d(\mathcal{L}_X\lambda' - \mathcal{L}_{X'}\lambda) = \delta_{([X,X']_{\mathfrak{X}},\mathcal{L}_X\lambda' - \mathcal{L}_{X'}\lambda)}A
\end{aligned}
$$

- non-abelian case: $[(X,\lambda),(X',\lambda')] = ([X,X']_{\mathfrak{X}}, \mathcal{L}_X\lambda' - \mathcal{L}_{X'}\lambda - [\lambda,\lambda']_{\mathfrak{g}})$
- properties: antisymmetric, Jacobi $\rightsquigarrow$ Lie bracket
- Jacobi $\Leftrightarrow$ $[u,[u',u'']] + [u',[u'',u]] + [u'',[u,u']] = 0 \Leftrightarrow [u,[u',u'']] = [[u,u'],u''] + [u',[u,u'']]$
- define vector bundle E as a direct sum of TM and the trivial bundle $M \times \mathfrak{g} =: \mathrm{ad}$

$$E := TM \oplus \mathrm{ad}, \qquad \Gamma(E) \ni (X,\lambda)$$

  sections of E correspond to (X,λ); hence $\Gamma(E)$ is a Lie algebra

$$[X+\lambda, X'+\lambda']_E = [X,X']_{\mathfrak{X}} + \mathcal{L}_X\lambda' - \mathcal{L}_{X'}\lambda - [\lambda,\lambda']_{\mathfrak{g}}$$

- if $f \in C^\infty(M)$ then

$$[X+\lambda, f(X'+\lambda')] = \mathcal{L}_X(fX') + \mathcal{L}_X(f\lambda') - f\mathcal{L}_{X'}\lambda - f[\lambda,\lambda']_{\mathfrak{g}} = f[X+\lambda, X'+\lambda'] + (Xf)(X'+\lambda')$$

  if define the vector bundle map (*anchor*) $\rho\colon E \to TM$, $\rho(X+\lambda) = X$, then

$$[u, fu']_E = f[u,u']_E + (\rho(u)f)u' \qquad \forall u,u' \in \Gamma(E)$$

- a *Lie algebroid* is a vector bundle $E \to M$, with a Lie bracket $[\,\cdot\,,\,\cdot\,]_E$ on $\Gamma(E)$ and a vector bundle map $\rho\colon E \to TM$ satisfying the last equation
- example: $E = TM \oplus \mathrm{ad}$
  - if $\mathfrak{g} = 0$ then TM, $[\,\cdot\,,\,\cdot\,]_{\mathfrak{X}}$, $\rho = \mathrm{id}$
  - if $M = \mathrm{pt}$ then $E = \mathfrak{g}$, $[\,\cdot\,,\,\cdot\,]_{\mathfrak{g}}$, $\rho = 0$ (Lie algebroid over a point is a Lie algebra)

String theory low energy effective action (universal sector)

$$S(g,B,\varphi) = \int_M e^{-2\varphi}(R\,\mathrm{vol}_g + 4\,d\varphi \wedge *d\varphi - \tfrac{1}{2}\,dB \wedge *dB) \qquad g \text{ metric}, B \in \Omega^2(M), \varphi \in C^\infty(M)$$

- analogue of the previous abelian theory, with connection replaced by a "higher connection"
- symmetries generated by $X \in \Gamma(TM)$, $\alpha \in \Omega^1(M)$

$$\delta_{(X,\alpha)}g = \mathcal{L}_X g, \qquad \delta_{(X,\alpha)}B = \mathcal{L}_X B + d\alpha, \qquad \delta_{(X,\alpha)}\varphi = \mathcal{L}_X\varphi$$

- as before, $(\delta_{(X,\alpha)}\delta_{(X',\alpha')} - \delta_{(X',\alpha')}\delta_{(X,\alpha)})B = \mathcal{L}_{[X,X']_{\mathfrak{X}}}B + d(\mathcal{L}_X\alpha' - \mathcal{L}_{X'}\alpha)$
- define vector bundle $E := TM \oplus T^*M$, sections of E correspond to $(X,\alpha) \equiv X + \alpha$
- produces bracket $[X+\alpha, X'+\alpha']_E = [X,X']_{\mathfrak{X}} + \mathcal{L}_X\alpha' - \mathcal{L}_{X'}\alpha$

- antisymmetric, but not Jacobi! solution: we can just as well write

$$(\delta_{(X,\alpha)}\delta_{(X',\alpha')} - \delta_{(X',\alpha')}\delta_{(X,\alpha)})B = \mathcal{L}_{[X,X']_{\mathfrak{X}}}B + d(\mathcal{L}_X\alpha' - \mathcal{L}_{X'}\alpha + d(\dots)),$$

  with $\dots$ an expression linear in X, α, X', α'

- choosing $\dots = di_{X'}\alpha$, we get the bracket $\boxed{[X+\alpha, X'+\alpha']_E = [X,X']_{\mathfrak{X}} + \mathcal{L}_X\alpha' - i_{X'}d\alpha}$
- not antisymmetric, but satisfies Jacobi $[u,[u',u'']_E]_E = [[u,u']_E,u'']_E + [u',[u,u'']_E]_E$
- again, defining $\rho\colon E \to TM$, $\rho(X+\alpha) = X$, we get $[u,fu']_E = f[u,u']_E + (\rho(u)f)u'$
- $TM \oplus T^*M$ carries more structure, can define natural inner product

$$\langle X+\alpha, X'+\alpha'\rangle := \alpha(X') + \alpha'(X)$$

- this is "preserved" by the bracket, since (setting $u = X+\alpha$, $u' = X'+\alpha'$, $u'' = X''+\alpha''$)

$$\begin{aligned}\langle [u,u']_E, u''\rangle + \langle u', [u,u'']_E\rangle &= \langle [X,X']_{\mathfrak{X}} + \mathcal{L}_X\alpha' - i_{X'}d\alpha, X''+\alpha''\rangle + \text{ }'\leftrightarrow''\\ &= i_{X''}\mathcal{L}_X\alpha' - i_{X''}i_{X'}d\alpha + i_{[X,X']_{\mathfrak{X}}}\alpha'' + i_{X'}\mathcal{L}_X\alpha'' - i_{X'}i_{X''}d\alpha + i_{[X,X'']_{\mathfrak{X}}}\alpha'\\ &= (i_{[X,X']_{\mathfrak{X}}}\alpha'' + i_{X'}\mathcal{L}_X\alpha'') + (i_{X''}\mathcal{L}_X\alpha' + i_{[X,X'']_{\mathfrak{X}}}\alpha') - (i_{X'}i_{X''} + i_{X''}i_{X'})d\alpha\\ &= \mathcal{L}_X i_{X'}\alpha'' + \mathcal{L}_X i_{X''}\alpha' = X(i_{X'}\alpha'' + i_{X''}\alpha') = \rho(u)\langle u', u''\rangle\end{aligned}$$

- the symmetric part of the bracket is $[u,u']_E + [u',u]_E = d(i_{X'}\alpha + i_X\alpha') = d\langle u,u'\rangle$
- a *Courant algebroid* is a vector bundle $E \to M$, with an ($\mathbb{R}$-bilinear) bracket $[\,\cdot\,,\,\cdot\,]_E$ on $\Gamma(E)$, a vector bundle map $\rho\colon E \to TM$, and a fibrewise inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ (not necessarily positive-definite) such that

$$[u,fu']_E = f[u,u']_E + (\rho(u)f)u', \qquad [u,[u',u'']_E]_E = [[u,u']_E,u'']_E + [u',[u,u'']_E]_E,$$

$$\rho(u)\langle u',u''\rangle = \langle [u,u']_E, u''\rangle + \langle u', [u,u'']_E\rangle, \qquad [u,u']_E + [u',u]_E = \rho^* d\langle u,u'\rangle$$

- note that in the last formula we have used $\rho^*\colon T^*M \to E^* \cong E$ (the last equality using the inner product), which in the above example coincides with the usual $T^*M \to TM \oplus T^*M$

3 Linear algebra intermezzo

- let V be a vector space with an inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ of signature (n,n)
- for a linear subspace $W \subset V$ define $W^\perp := \{x \in V \mid \langle x, W\rangle = 0\}$ (note that $(W^\perp)^\perp = W$)
- a linear subspace $W \subset V$ is called *isotropic* if $W \subset W^\perp$, i.e. $\langle W,W\rangle = 0$ ($\implies \dim W \leq n$)
- a linear subspace $W \subset V$ is called *Lagrangian* if $W = W^\perp$ ($\implies \dim W = n$)
- a linear subspace $W \subset V$ is called *coisotropic* if $W \supset W^\perp$ ($\implies \dim W \geq n$)
- if W, W' are Lagrangian and $W \cap W' = 0$, then the map $W' \to W^*$, $x \mapsto \langle x,\,\cdot\,\rangle|_W$ is an isomorphism
- Lagrangian = both isotropic & coisotropic

4 Courant algebroids

- examples:
  - $TM \oplus T^*M$ as above (*generalised tangent bundle*)
  - if $M = \mathrm{pt}$ then $E =$ Lie algebra with invariant inner product
- using the first two axioms (and suppresing the subscript E) we get

$$\begin{aligned}0 &= [u,[u',fu'']] - [[u,u'],fu''] - [u',[u,fu'']]\\ &= [u, f[u',u''] + (\rho(u')f)u''] - (u \leftrightarrow u') - f[[u,u'],u''] - (\rho([u,u'])f)u''\\ &= f[u,[u',u'']] + (\rho(u)f)[u',u''] + (\rho(u')f)[u,u''] + (\rho(u)\rho(u')f)u'' - (u \leftrightarrow u')\\ &- f[[u,u'],u''] - (\rho([u,u'])f)u'' = f([u,[u',u'']] - [[u,u'],u''] - [u',[u,u'']])\\ &+ (\rho(u)\rho(u') - \rho(u')\rho(u) - \rho([u,u'])f)u'' = ([\rho(u),\rho(u')]_{\mathfrak{X}} - \rho([u,u'])f)u''\end{aligned}$$

  and so the anchor, seen as a map $\Gamma(E) \to \Gamma(TM)$, is a homomorphism of algebras, i.e.

$$[\rho(u),\rho(u')]_{\mathfrak{X}} = \rho([u,u']_E)$$

- applying this to the last axiom, and using the antisymmetry of $[\,\cdot\,,\,\cdot\,]_{\mathfrak{X}}$, we get

$$\rho\rho^* d\langle u,u'\rangle = 0$$

  since any function can be (locally) written as $\langle u,u'\rangle$, for some u,u', we have $\rho\rho^* df = 0$ for any function; writing any 1-form α in coordinates

$$\rho\rho^*\alpha = \rho\rho^*(\alpha_i(x)dx^i) = \alpha^i(x)\rho\rho^* dx^i = 0,$$

  since ρ and ρ^* are vector bundle maps, and so we can conclude that

$$\rho \circ \rho^* = 0$$

- this condition is equivalent to the fact that $\ker\rho$ is coisotropic (at every point on M)
- it is also equivalent to the following: consider the following sequence of (vector bundle) maps

$$0 \to T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \to 0$$

  with the first and last map trivial; then composing any two subsequent maps/arrows gives zero, i.e. this is a *chain complex*

Exact Courant algebroids

- we say that a Courant algebroid is *exact* if this sequence of maps is exact, i.e. each time we encounter

$$\ldots \xrightarrow{\kappa} \ldots \xrightarrow{\lambda} \ldots$$

  we require that $\operatorname{im}\kappa = \ker\lambda$
- this translates to the conditions $0 = \ker\rho^*$, $\operatorname{im}\rho^* = \ker\rho$, and $\operatorname{im}\rho = TM$, respectively; equivalently, ρ^* is injective, $\operatorname{im}\rho^* = \ker\rho$, and ρ is surjective

  however, the first condition follows from the last one and hence can be dropped similarly, since for any Courant algebroid we have $\operatorname{im}\rho^* \subset \ker\rho$, the equality $\operatorname{im}\rho^* = \ker\rho$ is equivalent to the fact that the dimensions match:[1] $\operatorname{rk}(\operatorname{im}\rho^*) = \operatorname{rk}(\ker\rho)$

assuming surjectivity of ρ (and hence injectivity of ρ^*), we have

$$\operatorname{rk}(\operatorname{im}\rho^*) = \operatorname{rk}(T^*M) = \dim M, \quad \operatorname{rk}(\ker\rho) = \operatorname{rk}(E) - \operatorname{rk}(TM) = \operatorname{rk}(E) - \dim M$$

and so a Courant algebroid is exact iff ρ is surjective and $\operatorname{rk}(E) = 2\dim M$

in particular, for any exact Courant algebroid the subbundle $\ker\rho$ is Lagrangian

- example: the generalised tangent bundle is an exact Courant algebroid
- fact: for any exact Courant algebroid one can choose a global Lagrangian splitting of the exact sequence, i.e. there exists a vector bundle map $\tau\colon TM \to E$ such that $\rho\circ\tau = \mathrm{id}$ and the image of τ is Lagrangian (to prove this one uses the partition of unity, and some extra stuff)
- choosing such a map τ, we get a vector bundle map

$$\Phi\colon TM \oplus T^*M \to E, \qquad X + \alpha \mapsto \tau(X) + \rho^*\alpha$$

this is injective: if we have $u = X + \alpha$ such that $0 = \Phi(u) = \tau(X) + \rho^*\alpha$, then applying ρ we in particular get $0 = \rho(\tau(X)) = X$; we thus have $0 = \Phi(u) = \rho^*\alpha$, which by injectivity of ρ^* implies $\alpha = 0$

since the ranks of the bundles coincide, we get that Φ is in fact an isomorphism

let us therefore use Φ to identify E with $TM \oplus T^*M$ and let us check what induced structure $\langle\,\cdot\,,\,\cdot\,\rangle_\Phi$, ρ_Φ, and $[\,\cdot\,,\,\cdot\,]_\Phi$ we get on $TM \oplus T^*M$; first,

$$\langle X + \alpha, X' + \alpha'\rangle_\Phi := \langle\Phi(X+\alpha), \Phi(X'+\alpha')\rangle_E = \langle\tau(X) + \rho^*\alpha, \tau(X') + \rho^*\alpha'\rangle$$

since $\operatorname{im}\tau$ and $\operatorname{im}\rho^* = \ker\rho$ are both Lagrangian, this simplifies to

$$\langle X+\alpha, X'+\alpha'\rangle_\Phi = \langle\rho^*\alpha, \tau(X')\rangle + \langle\rho^*\alpha', \tau(X)\rangle = \langle\alpha, \rho\tau(X')\rangle + \langle\alpha', \rho\tau(X)\rangle = \alpha(X') + \alpha'(X)$$

by construction, we have $\rho_\Phi(X+\alpha) = \rho(\Phi(X+\alpha)) = X$; for the bracket, first note that $[\rho_\Phi(u), \rho_\Phi(u')]_{\mathfrak{X}} = \rho_\Phi([u,u']_\Phi)$ implies $[X, X']_{\mathfrak{X}} = \rho_\Phi([X+\alpha, X'+\alpha']_\Phi)$, i.e.

$$[X+\alpha, X'+\alpha']_\Phi = [X, X']_{\mathfrak{X}} + \text{(some 1-form expression)}$$

in particular

$$[X, X']_\Phi = [X, X']_\Phi + \Xi(X, X'), \qquad \Xi\colon \Gamma(TM)\times\Gamma(TM) \to \Gamma(T^*M)$$

using the fourth axiom for $u = X$, $u = X'$ we see that $\Xi(X,X') = -\Xi(X',X)$; the third axiom gives

$$0 = \Xi(X,X')(X'') + \Xi(X,X'')(X')$$

which implies that $\Xi(X,X')(X'')$ is completely antisymmetric in all three arguments; moreover, since it is $C^\infty(M)$-linear in the third argument, it has to be $C^\infty(M)$-linear in all three, and so we get that $\Xi(X,X')(X'') = H(X,X',X'')$ for some 3-form H; we can also obtain other useful formulas using the third axiom for various u, u', u'':

$$u = X,\ u' = X',\ u'' = \alpha \ \rightsquigarrow\ \mathcal{L}_X(\alpha(X')) = \alpha(\mathcal{L}_X X') + \langle X', [X,\alpha]_\Phi\rangle_\Phi \implies [X,\alpha]_\Phi = \mathcal{L}_X\alpha,$$

[1] the dimension of the fibres is called the *rank* of the vector bundle (denoted here by rk)

$$u=\alpha,\ u'=\alpha',\ u''=X \quad \rightsquigarrow \quad 0=\langle[\alpha,\alpha']_\Phi,X\rangle_\Phi \quad \Longrightarrow \quad [\alpha,\alpha']_\Phi=0$$

the last axiom implies

$$[\alpha,X]_\Phi=-[X,\alpha]_\Phi+d\langle\alpha,X\rangle_\Phi=-\mathcal{L}_X\alpha+di_X\alpha=-i_Xd\alpha$$

putting things together we have

$$[X+\alpha,X'+\alpha']_\Phi=[X,X']_{\mathfrak{X}}+\mathcal{L}_X\alpha'-i_{X'}d\alpha+H(X,X',\,\cdot\,)$$

note that in terms of τ we have $H(X,X',X'')=\langle[\tau(X),\tau(X')],\tau(X'')\rangle$
using Jacobi for three vectors, we have

$$\begin{aligned}
0&=[X,[X',X'']_\Phi]_\Phi-[[X,X']_\Phi,X'']_\Phi-[X',[X,X'']_\Phi]_\Phi\\
&=[X,[X',X'']_{\mathfrak{X}}+i_{X''}i_{X'}H]_\Phi-(X\leftrightarrow X')-[[X,X']_{\mathfrak{X}}+i_{X'}i_XH,X'']_\Phi\\
&=[X,[X',X'']_{\mathfrak{X}}]_{\mathfrak{X}}+i_{[X',X'']_{\mathfrak{X}}}i_XH+\mathcal{L}_Xi_{X''}i_{X'}H\\
&-[X',[X,X'']_{\mathfrak{X}}]_{\mathfrak{X}}-i_{[X,X'']_{\mathfrak{X}}}i_{X'}H-\mathcal{L}_{X'}i_{X''}i_XH\\
&-[[X,X']_{\mathfrak{X}},X'']_{\mathfrak{X}}-i_{X''}i_{[X,X']_{\mathfrak{X}}}H+i_{X''}di_{X'}i_XH\\
&=(i_{[X',X'']_{\mathfrak{X}}}i_X+\mathcal{L}_Xi_{X''}i_{X'}-i_{[X,X'']_{\mathfrak{X}}}i_{X'}-\mathcal{L}_{X'}i_{X''}i_X-i_{X''}i_{[X,X']_{\mathfrak{X}}}+i_{X''}di_{X'}i_X)H\\
&=(i_{X''}\mathcal{L}_{X'}i_X+i_{X''}\mathcal{L}_Xi_{X'}-(i_{X''}\mathcal{L}_Xi_{X'}-i_{X''}i_{X'}\mathcal{L}_X)+i_{X''}di_{X'}i_X)H\\
&=i_{X''}(\mathcal{L}_{X'}i_X+i_{X'}\mathcal{L}_X+di_{X'}i_X)H=i_{X''}(i_{X'}di_X+i_{X'}\mathcal{L}_X)H=i_{X''}i_{X'}i_XdH
\end{aligned}$$

and so H is closed; it is straightforward to check that, assuming $dH=0$, all the Courant algebroid axioms are satisfied (there is also a more sneaky argument, see below); to summarise, our starting exact Courant algebroid can be recast in the following form:

$$TM\oplus T^*M,\quad \rho_\Phi(X+\alpha)=0,\quad \langle X+\alpha,X'+\alpha'\rangle_\Phi=\alpha(X')+\alpha'(X),$$

$$[X+\alpha,X'+\alpha']_\Phi=[X,X']_{\mathfrak{X}}+\mathcal{L}_X\alpha'-i_{X'}d\alpha+H(X,X',\,\cdot\,)\qquad (H\in\Omega^3(M),dH=0)$$

- recall now that all this depended on a choice of τ, which led to a specific Φ; what happens when we take a different Lagrangian splitting τ'? note that the only thing that can change is the 3-form H entering the bracket

  in order to proceed, let us look at the algebroid $E=TM\oplus T^*M$, ρ_Φ, $\langle\,\cdot\,,\,\cdot\,\rangle_\Phi$, $[\,\cdot\,,\,\cdot\,]_\Phi$ and omit the subscript Φ; note that in this form we have a natural Lagrangian splitting $TM\to TM\oplus T^*M$ of the exact sequence; suppose we take instead a general Lagrangian splitting τ'; then

  $$\tau'(X)=X+\lambda(X)\in TM\oplus T^*M,$$

  where $\lambda\colon TM\to T^*M$ is a bundle map; isotropy implies that

  $$0=\langle X+\lambda(X),X'+\lambda(X')\rangle=\lambda(X)(X')+\lambda(X')(X)$$

  and so we have $\lambda(X)(X')=B(X,X')$ for some $B\in\Omega^2(M)$ (conversely, any B gives a Lagrangian splitting); the 3-form H corresponding to this splitting is then calculated to be

  $$\begin{aligned}
  H'(X,X',X'')&=\langle[\tau'(X),\tau'(X')],\tau'(X'')\rangle=\langle[X+i_XB,X'+i_{X'}B],X''+i_{X''}B\rangle\\
  &=i_{[X,X']_{\mathfrak{X}}}i_{X''}B+i_{X''}(\mathcal{L}_Xi_{X'}B-i_{X'}di_XB+i_{X'}i_XH)\\
  &=i_{X''}(i_{X'}\mathcal{L}_XB-i_{X'}di_XB+i_{X'}i_XH)=i_{X''}i_{X'}i_X(H+dB)
  \end{aligned}$$

  changing the splitting thus leads to a shift of H by dB; we have thus proved the following remarkable result

- Theorem (Ševera): Exact Courant algebroids over a given M are classified by $H^3(M)$. Choosing a specific Lagrangian splitting τ of the exact sequence, we obtain a concrete three-form representative given by $H(X, X', X'') = \langle[\tau(X), \tau(X')], \tau(X'')\rangle$, and we can write the Courant algebroid structure as

$$TM \oplus T^*M, \quad \rho(X+\alpha) = 0, \quad \langle X+\alpha, X'+\alpha'\rangle = \alpha(X') + \alpha'(X),$$

$$[X+\alpha, X'+\alpha']_E = [X, X']_{\mathfrak{X}} + \mathcal{L}_X\alpha' - i_{X'}d\alpha + H(X, X', \,\cdot\,)$$

(we will call the special case with $H = 0$ the *standard Courant algebroid*)

Dirac structures

- a subbundle $L \subset E$ of a Courant algebroid is called a *Dirac structure* if it is Lagrangian (at every point) and involutive, i.e.

$$[\Gamma(E), \Gamma(E)]_E \subset \Gamma(E)$$

- if L is a Dirac structure in the standard Courant algebroid satisfying $L \cap T^*M = 0$ then

$$L = \operatorname{graph}\omega := \{X + \omega(X, \,\cdot\,) \mid X \in TM\},$$

for some tensor $\omega \in \mathcal{T}^0_2(M)$; as we saw above, the Lagrangianity conditions says that ω is antisymmetric; the involutivity means that for every X and X' there exists X'' such that

$$[X + i_X\omega, X' + i_{X'}\omega]_E = X'' + i_{X''}\omega$$

this fixes $X'' = [X, X']_{\mathfrak{X}}$, and the condition then becomes

$$\mathcal{L}_X i_{X'}\omega - i_{X'}di_X\omega = i_{[X,X']_{\mathfrak{X}}}\omega$$

putting everything to one side, we get

$$0 = (\mathcal{L}_X i_{X'} - i_{X'}di_X - i_{[X,X']_{\mathfrak{X}}})\omega = (i_{X'}\mathcal{L}_X - i_{X'}di_X)\omega = i_{X'}i_X d\omega$$

and so we see that Dirac structures in a standard Courant algebroid, which are transverse to T^*M, correspond to closed 2-forms; this provides a link with symplectic geometry (though note that we do not get here that ω is nondegenerate)

- similarly, if L is a Dirac structure in the standard Courant algebroid satisfying $L \cap TM = 0$ then it is given by a graph of a bivector field π; the involutivity is equivalent to π being a Poisson structure

Generalised complex structures

- starting from a Courant algebroid $E \to M$, we can complexify the fibers to get a smooth complex vector bundle $E^{\mathbb{C}} \to M$; the bracket and the inner product naturally extend to $\mathbb{C}$-bilinear structures on this bundle; in particular we can again define a Dirac structure on $E^{\mathbb{C}}$, which we then call a *complex Dirac structure*; note also that we can use complex conjugation on $E^{\mathbb{C}}$

- a *generalised complex structure* is a complex Dirac structure $L \subset E^{\mathbb{C}}$ such that $L \cap \bar{L} = 0$

- a symplectic structure defines a generalised complex structure $L := \operatorname{graph}(i\omega)$

- a complex structure defines a generalised complex structure $L := T^{(1,0)}M \oplus T^{*(0,1)}M$

Generalised metrics

- we will specialise here to the Riemannian (i.e. positive-definite) setup; it is however not difficult to adjust the definitions to accommodate the Lorentzian (or orther) cases as well
- a *generalised metric* on a Courant algebroid is a subbundle $V_+ \subset E$ which is maximally positive definite, i.e. the following two conditions hold
  - the inner product restricted to V_+ is positive definite
  - the inner product restricted to the orthogonal complement of V_+, denoted V_-, is negative definite
- let V_+ be a generalised metric on an exact Courant algebroid; fix a Lagrangian splitting $\tau\colon TM \to E$; this gives an identification $E \cong TM \oplus T^*M$ and a particular 3-form H; note that $V_+ \cap T^*M = 0$ since every vector in T^*M has zero norm, while every non-zero vector in V_+ has non-zero norm; thus we can write

$$V_+ = \mathrm{graph}(e) = \{X + e(X,\ \cdot\ ) \mid X \in TM\}$$

for some $e \in \mathcal{T}^0_2(M)$; decompose now e into a symmetric and antisymmetric part

$$e = g + b$$

since $\langle X + e(X,\ \cdot\ ), X' + e(X',\ \cdot\ )\rangle = e(X,X') + e(X',X) = 2g(X,X')$, the metric on V_+ coincides (up to a multiple) with g, the latter is a Riemannian metric on M; note that we have produced three tensors: g, b, H

now choosing a different Lagrangian splitting τ' leads to a different triplet; from the previous discussion we know that H gets replaced by $H+dB$, for some $B \in \Omega^2(M)$; to see how $e = g+b$ changes, we need to decompose elements in $V_+ \subset TM \oplus T^*M$ into T^*M and

$$\mathrm{im}(\tau') = \mathrm{graph}(B) = \{X + B(X,\ \cdot\ ) \mid X \in TM\}$$

this is readily done, as

$$V_+ = \{X + e(X,\ \cdot\ ) \mid X \in TM\} = \{(X + B(X,\ \cdot\ )) + (e(X,\ \cdot\ ) - B(X,\ \cdot\ )) \mid X \in TM\}$$

and so we see that e gets replaced by $e - B$, i.e.

$$g \rightsquigarrow g, \qquad b \rightsquigarrow b - B$$

so it is only the metric g and the combination $\hat{H} := H + db$ that is independent of the choice of splitting; we thus have a one-to-one correspondence

$$\text{generalised metric on an exact Courant algebroid} \;\leftrightarrow\; \text{Riemannian metric \& closed 3-form}$$

an equivalent way to look at things is to note that there exists a unique isotropic splitting such that V_+ becomes the graph of a symmetric tensor (i.e. it has $b = 0$)